MA 3053 Practice Exam 1 Solutions

1.) let $f: X \to Y$, $g: Y \to Z$. Prove that if $g \circ f$ is 1-1, then so is f.

pf: 1st note $dom(g) = Y$ and $ran(f) \subseteq Y$ so $g \circ f$ is well-def. Then s'pose $f(x_1) = f(x_2)$ for $x_1, x_2 \in X$. then $\Rightarrow g(f(x_1)) = g(f(x_2)) \Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2) \Rightarrow x_1 = x_2$ since $g \circ f$ is 1-1. $\Rightarrow f$ is 1-1. $\Box$

2.) let $f: X \to Y$. Given fncs $g, h: W \to X$ such that whenever $f \circ g = f \circ h$, then $g = h$. Prove f is 1-1.

pf: Consider $g(w_1)$ and $h(w_2)$ for $w_1, w_2 \in W$. Since $g, h: W \to X \Rightarrow g(w_1), h(w_2) \in X$ set $x_1 = g(w_1)$ and $x_2 = h(w_2)$. and consider $f(x_1) = f(x_2)$. $\Rightarrow f(g(w_1)) = f(h(w_2))$ $\Rightarrow (f \circ g)(w_1) = (f \circ h)(w_2)$ but by hypothesis $\Rightarrow g = h \Rightarrow g(w_1) = h(w_2) \Rightarrow x_1 = x_2$ so f is 1-1. $\Box$

3.) let $f: X \to Y$ and $P_\alpha \subseteq Y$, for all $\alpha \in A$. Prove $f^{-1}(\bigcup_{\alpha \in A} P_\alpha) = \bigcup_{\alpha \in A} f^{-1}(P_\alpha)$.

pf: (a) let $x \in f^{-1}(\bigcup_{\alpha \in A} P_\alpha) \Rightarrow f(x) \in \bigcup_{\alpha \in A} P_\alpha \Rightarrow f(x) \in P_\alpha$ for some $\alpha \in A$. $\Rightarrow x \in f^{-1}(P_\alpha)$ for some $\alpha \in A$. $\Rightarrow x \in \bigcup_{\alpha \in A} f^{-1}(P_\alpha)$. $\Rightarrow f^{-1}(\bigcup_{\alpha \in A} P_\alpha) \subseteq \bigcup_{\alpha \in A} f^{-1}(P_\alpha)$.

(b) let $x \in \bigcup_{\alpha \in A} f^{-1}(P_\alpha)$. $\Rightarrow x \in f^{-1}(P_\alpha)$ for some $\alpha \in A$. $\Rightarrow f(x) \in P_\alpha$ for some $\alpha \in A$ $\Rightarrow f(x) \in \bigcup_{\alpha \in A} P_\alpha \Rightarrow x \in f^{-1}(\bigcup_{\alpha \in A} P_\alpha) \Rightarrow \bigcup_{\alpha \in A} f^{-1}(P_\alpha) \subseteq f^{-1}(\bigcup_{\alpha \in A} P_\alpha)$.

so by (a), (b) $\Rightarrow f^{-1}(\bigcup_{\alpha \in A} P_\alpha) = \bigcup_{\alpha \in A} f^{-1}(P_\alpha)$ $\Box$

4.) let $f: X \to Y$, $P_\alpha \subseteq X$ for all $\alpha \in A$. Prove $f(\bigcup_{\alpha \in A} P_\alpha) = \bigcup_{\alpha \in A} f(P_\alpha)$.

pf: (a) let $f(x) \in f(\bigcup_{\alpha \in A} P_\alpha) \Rightarrow x \in \bigcup_{\alpha \in A} P_\alpha \Rightarrow x \in P_\alpha$ for some $\alpha \in A$. $\Rightarrow f(x) \in f(P_\alpha)$ for some $\alpha \in A$ $\Rightarrow f(x) \in \bigcup_{\alpha \in A} f(P_\alpha)$. $\Rightarrow f(\bigcup_{\alpha \in A} P_\alpha) \subseteq \bigcup_{\alpha \in A} f(P_\alpha)$.

(b) let $f(x) \in \bigcup_{\alpha \in A} f(P_\alpha)$. $\Rightarrow f(x) \in f(P_\alpha)$ for some $\alpha \in A \Rightarrow x \in P_\alpha$ for some $\alpha \in A$ $\Rightarrow x \in \bigcup_{\alpha \in A} P_\alpha$ $\Rightarrow f(x) \in f(\bigcup_{\alpha \in A} P_\alpha) \Rightarrow \bigcup_{\alpha \in A} f(P_\alpha) \subseteq f(\bigcup_{\alpha \in A} P_\alpha)$

so by (a), (b) $\Rightarrow f(\bigcup_{\alpha \in A} P_\alpha) = \bigcup_{\alpha \in A} f(P_\alpha)$. $\Box$

5.) let $\sim$ be a relation on $X = \mathbb{Z} \times \mathbb{Z}$. by $(a,b) \sim (c,d)$ iff $a + d = b + c$. Prove $\sim$ is an equiv rel.

pf: 1st Refl: consider $(a,b) \sim (a,b)$ is true iff $a + b = b + a$ but addition is commutative so true. 2nd symmetric: let $(a,b) \sim (c,d) \Rightarrow a + d = b + c \Rightarrow c + b = d + a$ as add. is comm. $\Rightarrow (c,d) \sim (a,b)$ so sym. 3rd let $(a,b) \sim (c,d)$ and $(c,d) \sim (x,y)$ so $\Rightarrow a + d = b + c$ and $c + y = d + x \Rightarrow c = d + x - y$ $\Rightarrow a + d = b + d + x - y \Rightarrow a + y = b + x \Rightarrow (a,b) \sim (x,y)$. so transitive. $\Box$

6.) let $f: X \to Y$. let $\sim$ be a relation on X by $x \sim y$ iff $f(x) = f(y)$. Prove $\sim$ is equivalence relation

pf:

1st refl. Consider ~~x~~ $x \sim x$ is true iff $f(x) = f(x)$. but f is a fnc so $f(x) = f(x)$ always true. 2nd let $x \sim y \Rightarrow f(x) = f(y) \Rightarrow f(y) = f(x) \Rightarrow y \sim x$. so $\sim$ is symm. 3rd let $x \sim y$ and $y \sim z \Rightarrow f(x) = f(y)$ and $f(y) = f(z) \Rightarrow f(x) = f(y) = f(z)$ so $f(x) = f(z) \Rightarrow x \sim z$. So $\sim$ is trans. $\Box$

7.) Let $\mathcal{F}$ be a family of sets and $\preceq$ be a rel. on $\mathcal{F}$ by $X \preceq Y$ iff $X \subseteq Y$. Prove $\preceq$ is a partial order on $\mathcal{F}$.

pf:

Since $X \subseteq X$ is always true have $X \preceq X$. so ~~a~~ $\preceq$ is ~~refl~~ refl. Next let $X \preceq Y$ and $Y \preceq X \Rightarrow X \subseteq Y$ and $Y \subseteq X$, by def of set containment $\Rightarrow X = Y$. so $\preceq$ is antisymm. Finally let $X \preceq Y$ and $Y \preceq Z \Rightarrow X \subseteq Y$ and $Y \subseteq Z \Rightarrow X \subseteq Y \subseteq Z \Rightarrow X \subseteq Z$ so $X \preceq Z \Rightarrow \preceq$ is trans. $\Box$.

8.) let $\preceq$ be a rel. on $\mathbb{R}^n$ as follows, for $x \neq y$ let $k \in \mathbb{N}^+$ s.t. $a_k \neq b_k$ w/ $x = (a_1, \dots, a_n)$, $y = (b_1, \dots, b_n)$ then $x \preceq y$ iff $a_k < b_k$. Show $\preceq$ is a partial order on $\mathbb{R}^n$.

pf:

refl holds by default b/c can only compare distinct elmts. (why?) Now let $X \preceq Y$ and $Y \preceq X \Rightarrow a_k < b_k$ and $b_j < a_j$ for ~~some~~ $k, j \in \mathbb{N}^+$ ~~$\exists$ there is a~~ let $l = \min(k, j)$ $\Rightarrow a_l < b_l$ and $b_l < a_l$ contradiction $\Rightarrow a_k = b_k$ ~~$\not$~~ for all $k = 1, \dots, n \Rightarrow X = Y$. lastly let $X \preceq Y$, $Y \preceq Z \Rightarrow$ there is $k, j \in \mathbb{N}^+$ s.t. $a_k < b_k$ and $b_j < c_j$ w/ $Z = (c_1, \dots, c_n)$ let $l = \max(k, j) \Rightarrow a_l < b_l < c_l \Rightarrow a_l < c_l \Rightarrow X \preceq Z$. So trans. $\Box$

9.) what are the multi/add tables for $\mathbb{Z}/14\mathbb{Z}$

pf:

notice upon multi/add we divide by 14. find remainder. so

+	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	0
2	2	3	4	5	6	7	8	9	10	11	12	13	14	0	1
3	3	4	5	6	7	8	9	10	11	12	13	14	0	1	2
4	4	5	6	7	8	9	10	11	12	13	14	0	1	2	3
5	5	6	7	8	9	10	11	12	13	14	0	1	2	3	4
6	6	7	8	9	10	11	12	13	14	15	1	2	3	4	5
7	7	8	9	10	11	12	13	14	0	~~1~~	2	3	4	5	6
8	8	9	10	11	12	13	14	0	1	~~2~~	3	4	5	6	7
9	9	10	11	12	13	14	0	1	2	3	4	5	6	7	8
10	10	11	12	13	14	0	1	2	3	4	5	6	7	8	9
11	11	12	13	14	0	1	2	3	4	5	6	7	8	9	10
12	12	13	14	0	1	2	3	4	5	6	7	8	9	10	11
13	13	14	0	1	2	3	4	5	6	7	8	9	10	11	12
14	14	0	1	2	3	4	5	6	7	8	9	10	11	12	13

Similar for multiplication.